JOURNAL OF COMPUTATIONAL PHYSICS 30, 222–237 (1979)

# Nonreflecting Boundary Conditions for Nonlinear Hyperbolic Systems

G. W. HEDSTROM

*Lawrence Livermore Laboratory, University of California, Livermore, California 94550*

Received October 28, 1977; revised April 21, 1978

Consider a nonlinear hyperbolic system $v_t + A(v)v_x = 0$ for $x > 0$ and $t > 0$. Suppose that the boundary $x = 0$ has been introduced only in order to limit the size of a computational problem. Suppose also that on physical grounds we know that no waves cross the boundary from the region $x < 0$. We need a boundary condition at $x = 0$ which expresses this fact. If our problem has no strong outgoing shocks, we may use the condition that at $x = 0$ the solution $v$ lies in the manifold generated by the Riemann invariants of the outgoing characteristics. For the equations of gas dynamics with an outflow boundary at $x = 0$ this condition may be written $c_v\gamma a p_t + c_v\gamma p u_t + a p S_t = 0$, where $a$ is the sound speed.

## 1. INTRODUCTION

In this paper we obtain a nonreflecting boundary condition at $x = 0$ for a nonlinear hyperbolic system

$$u_t + A(u)\,u_x = 0 \qquad ((x, t) \text{ in } \Omega), \tag{1.1}$$
$$u(x, 0) = g(x).$$

Here $\Omega$ is the quadrant $x > 0$, $t > 0$. We want nonreflecting boundary conditions, for example, if the boundary $x = 0$ is present only to make a computational problem smaller and if the phenomenon to be modeled has no waves entering $\Omega$ at the boundary $x = 0$. Note, though, that the phrase "no waves entering $\Omega$ at the boundary $x = 0$," says more than might be expected. For one thing, our boundary condition gives no reflection from a simple outgoing compression wave. We know [1] that a compression wave will develop a shock. As the shcok forms, waves propagate along the other characteristics since the Riemann invariants are not constant across a shock [2]. So, physically, an outgoing compression wave may well give rise to incoming waves later. For another thing, our boundary condition gives no reflection from a slow outgoing simple wave followed by a fast one. Physically, however, when the fast wave overtakes the slow one, there may well be an echo. We show in the Appendix that this is the case in gasdynamics. See also [3, p. 180]. Finally, we remark that because our boundary condition is based on the Riemann invariants, it produces a reflection from an outgoing shock. This reflection is very weak, though, for a weak shock. In order to get no reflection from an outgoing shock, we may well want to engage in shock tracking.



0021-9991/79/020222-16$02.00/0

Others have developed nonreflecting boundary conditions for linear hyperbolic systems. See in particular the paper of Engquist and Majda [4], where nonreflecting boundary conditions are obtained for linear systems in several space dimensions. The paper [4] also contains a bibliography of earlier works on the subject. When restricted to one space dimension and applied to the linearized version of (1.1),

$$u_t + A_0 u_x = 0 \qquad ((x, t) \text{ in } \Omega),$$

with constant matrix $A_0$, the nonreflecting boundary condition of [4] becomes "the characteristic variavles corresponding to incoming characteristic curves are constant." It is interesting that our nonreflecting boundary condition may be phrased in the same way, even though our derivation is based on a point of view taking the nonlinearity into account. We have not obtained extensions to nonlinear systems of the approximate nonreflecting boundary conditions of [4] for hyperbolic systems in several space dimensions.

We develop our nonreflecting boundary condition in the next section, and we apply it to the equations of gasdynamics in Section 3. We show the results of some computations in Section 4.

## 2. Nonreflecting Boundary Conditions

Let us order the eigenvalues $\lambda_j$ of $A(u)$ so that

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n .$$

We assume that the $\lambda_j$ are real and distinct, thus making the system (1.1) strictly hyperbolic. Suppose that $\lambda_1, \lambda_2, ..., \lambda_m$ are negative at $x = 0$ and that $\lambda_{m+1}, ..., \lambda_n$ are positive at $x = 0$. Thus, we have $m$ outgoing characteristics and $n$-$m$ incoming characteristics. The problem of reflection is interesting only if $1 \leqslant m < n$, so we assume that this is the case. We denote the left eigenvectors of $A(u)$ by $l_j$,

$$l_j A(u) = \lambda_j l_j \qquad (j = 1, 2, ..., n).$$

With this background our nonreflecting boundary condition is as follows.

THEOREM. *The condition at $x = 0$*

$$l_j \cdot \frac{\partial u}{\partial t} = 0 \qquad (m < j \leqslant n) \tag{2.1}$$

*gives no waves coming into $\Omega$ from the boundary $x = 0$ if there are only simple waves going out. If a shock of strength $\epsilon$ leaves $\Omega$, condition (2.1) at $x = 0$ produces an incoming wave of strength $O(\epsilon^3)$.*

*Remark.* The condition, "the characteristic variables corresponding to incoming characteristic curves are constant", may be written

$$l_j \cdot u = \text{const} \qquad (m < j \leqslant n). \tag{2.2}$$

In the linear case with constant matrix $A_0$, the $l_j$ are constant so that (2.2) and (2.1) are equivalent.

*Proof.* If there is a simple wave leaving $\Omega$, the corresponding Riemann invariants are constant [5, pp. 86–91]. Geometrically, we may express this fact by saying that in a simple wave corresponding to an eigenvalue $\lambda_k$ the values of $u$ lie on a curve $\Gamma_k$ in $n$-dimensional $u$-space. Furthermore, the right eigenvector $r_k$,

$$A(u)\, r_k = \lambda_k r_k ,$$

is tangent to $\Gamma_k$ [5, p. 88]. That is, along $\Gamma_k$ we have

$$du = c_k r_k . \tag{2.3}$$

Suppose that at $x = 0$ the vector $u$ starts at $t = 0$ with a value $u(0, 0)$ and that simple waves corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_m$ in succession cross the boundary. In $u$-space this corresponds to families of curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ in that order generating an $m$-dimensional manifold $M$. (We show in the Appendix that the order can make a difference.) That the manifold $M$ is $m$-dimensional follows from the linear independence of the eigenvectors $r_k$.

We now show that the condition, $u$ is in $M$, is equivalent to (2.1). In terms of differentials $ds_k$ along $\Gamma_k$ it follows from (2.3) that $u$ is in $M$ if and only if

$$du = \sum_{k=1}^{m} r_k\, ds_k . \tag{2.4}$$

The vectors $l_j$ are orthogonal to the $r_k$ for $j \neq k$ since

$$\begin{aligned} (l_j A)\, r_k &= \lambda_j l_j \cdot r_k , \\ l_j (A r_k) &= \lambda_k l_j \cdot r_k . \end{aligned}$$

Hence, if (2.4) holds, then

$$l_j \cdot du = 0 \qquad (m < j \leqslant n). \tag{2.5}$$

A simple vector-space argument, based on the fact that

$$r_1, r_2, \ldots, r_m, l_{m+1}, \ldots, l_n$$

forms a basis, shows that if (2.5) holds, then (2.4) is true. Since $du = (\partial u/\partial t)\, dt$ on the boundary $x = 0$, we have shown that $u$ is in $M$ if and only if (2.1) holds. This proves that (2.1) is nonreflecting for outgoing simple waves.

It remains to determine how much an outgoing shock is reflected by the boundary condition (2.1). The boundary condition (2.1) was obtained by assuming the constancy of the Riemann invariants. It is known [2] that across a shock of magnitude $\epsilon$ the Riemann invariants change by an amount $O(\epsilon^3)$. That is, the point $u$ moves off the manifold $M$ a distance $O(\epsilon^3)$, so that (2.4) is replaced by

$$du = \sum_{k=1}^{n} r_k \, ds_k ,$$
$$| r_k | \, ds_k = O(\epsilon^3) \qquad (m < k \leqslant n).$$

If we impose the boundary condition (2.1), or equivalently (2.4), we get reflections into $\Omega$ arising from the neglected terms $r_k \, ds_k$ with $k > m$. Thus, the reflections have magnitude $O(\epsilon^3)$. This proves the theorem.

## 3. Application to Gasdynamics

In one space dimension the equations of gasdynamics may be written [3, p. 28] in terms of the variables $\rho$, $u$, and $S$ in the form

$$\begin{aligned} \rho_t + u\rho_x + \rho u_x &= 0, \\ u_t + uu_x + \rho^{-1} p_x &= 0, \\ S_t + uS_x &= 0. \end{aligned}$$

Here, $\rho$ is the density, $u$ the velocity, $S$ the specific entropy, and $p$ the pressure. In the case of a polytropic ideal gas we have [3, pp. 6, 7]

$$p = \rho^\gamma(\gamma - 1) \exp\{(S - S_0)/c_v\} \tag{3.1}$$

with constants $\gamma$, $S_0$, and $c_v$. Thus, the matrix $A$ of (1.1) takes the form

$$A = \begin{pmatrix} u & \rho & 0 \\ \gamma p/\rho^2 & u & p/(\rho c_v) \\ 0 & 0 & u \end{pmatrix}. \tag{3.2}$$

In terms of the sound speed

$$a = (\gamma p/\rho)^{1/2} \tag{3.3}$$

the eigenvalues of $A$ are

$$\begin{aligned} \lambda_1 &= u - a, \\ \lambda_2 &= u, \\ \lambda_3 &= u + a. \end{aligned}$$

We assume that at the boundary $x = 0$ we have subsonic flow $| u | < a$, so that the reflection problem is nontrivial. Then $\lambda_1 < 0$ and $\lambda_3 > 0$. If $\lambda_2 > 0$, the boundary $x = 0$ is an inflow; if $\lambda_2 < 0$, it is an outflow.

The left eigenvectors for (2.1) are

$$l_2 = (0, 0, 1), \tag{3.4}$$

$$l_3 = (a, \rho, a\rho/(c_v\gamma)). \tag{3.5}$$

If $x = 0$ is an outflow boundary, the nonreflecting boundary condition (2.1) becomes

$$a\rho_t + \rho u_t + a\rho/(c_v\gamma)\, S_t = 0. \tag{3.6}$$

If $x = 0$ is an inflow boundary, the nonreflecting boundary condition (2.1) becomes (3.6) together with

$$S_t = 0.$$

Thus, the nonreflecting boundary conditions at an inflow $x = 0$ simplify to

$$a\rho_t + \rho u_t = 0, \tag{3.7}$$

$$S_t = 0. \tag{3.8}$$

It is clear that condition (3.6) may be written in terms of $\rho$, $u$, and $T$, the temperature. In fact, using the relations [3, pp. 4, 6]

$$c_v\, dT = T\, dS - pd(1/\rho),$$

$$p = \rho RT,$$

we easily obtain

$$T\, dS = c_v\, dT - (RT/\rho)\, d\rho.$$

Hence, since [3, pp. 6, 7]

$$R/c_v = \gamma - 1,$$

we see that (3.6) may be written

$$(T/\rho)\, \rho_t + (\gamma T/a)\, u_t + T_t = 0 \tag{3.9}$$

for nonreflection at an outflow $x = 0$.

## 4. Examples

We now report on some computational exaples. We used the Lax–Wendroff method [6, pp. 300–304] to solve the equations of gasdynamics written in Eulerian coordinates in conservation-law form

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2)_x + p_x &= 0, \\ e_t + (ue + up)_x &= 0, \qquad\qquad (0 < x < 1, t > 0), \\ T &= (e/\rho - u^2/2)/c_v\,, \\ p &= \rho RT. \end{aligned}$$

In these examples the flow is subsonic and $u > 0$, so that $x = 0$ is an inflow and $x = 1$ an outflow. Nonreflecting boundary conditions are used at both ends.

Because the Lax–Wendroff method requires values of all variables at both boundaries, we have to supplement the nonreflecting boundary conditions with some sort of extrapolation. We choose to extrapolate the characteristic variables along the outgoing chracteristics. Note that we already found the charadteristic variables when we obtained the nonreflecting boundary conditions. Thus, at the inflow $x = 0$ we use conditions (3.7) and (3.10) together with differentiation along the outgoing characteristic

$$(T/\rho)\,\rho_t - (\gamma T/a)\,u_t + T_t = (a - u)((T/\rho)\,\rho_x - (\gamma T/a)\,u_x + T_x).$$

This gives us three equations for $\rho_t$, $t_t$, and $T_t$, and with the notion

$$g = (a - u)((T/\rho)\,\rho_x - (\gamma T/a)\,u_x + T_x)$$

we easily obtain

$$\rho_t = g\rho/(2\gamma T), \tag{4.1}$$

$$u_t = -ag/(2\gamma T), \tag{4.2}$$

$$T_t = (\gamma - 1)\,g/(2\gamma). \tag{4.3}$$

In our implementation we determine $\rho$, $u$, and $T$ (and hence also $\rho u$ and $e$) at $x = 0$ and $t = t_{n+1}$ by discretizing (4.1)–(4.3) as follows. The coefficients are evaluated at $x = 0$ and $t = t_n$, and derivatives are differenced in the forward direction, so that

$$\rho_t \approx (\rho_0^{n+1} - \rho_0^{n})/\Delta t,$$

$$\rho_x \approx (\rho_1^{n} - \rho_0^{n})/\Delta x.$$

Similarly, at the outflow boundary $x = 1$ the nonreflecting boundary condition

$$(T/\rho)\,\rho_t - (\gamma T/a)\,u_t + T_t = 0$$

is supplemented by differentiation along the outgoing characteristics

$$(T/\rho)\,\rho_t + (\gamma T/a)\,u_t + T_t = -(a + u)(T/\rho)\,\rho_x + (\gamma T/a)\,u_x + T_x),$$

$$(T/\rho)\,\rho_t - (\gamma - 1)\,T_t = -u((T/\rho)\,\rho_x - (\gamma - 1)\,T_x),$$

to give a system of three equations for $\rho_t$, $u_t$, and $T_t$.

We remark that if equations (4.1)–(4.3) are replaced by expressions in terms of the fundamental variables $\rho$, $\rho u$, and $e$, the equations become considerably more complex. Formally it does not matter which set of variables we use, but once we take finite differences, the discretization error depends on the variables used.

Figures 1–6 show the results of a shock-tube computation compared with the exact solution. The pressure is not shown, but initially the pressure is ten times as high on the left as it is on the right. Figures 1 and 2 show the solution before any wave hits a boundary. In Figs. 3 and 4 the shock has hit the right-hand boundary and pro-

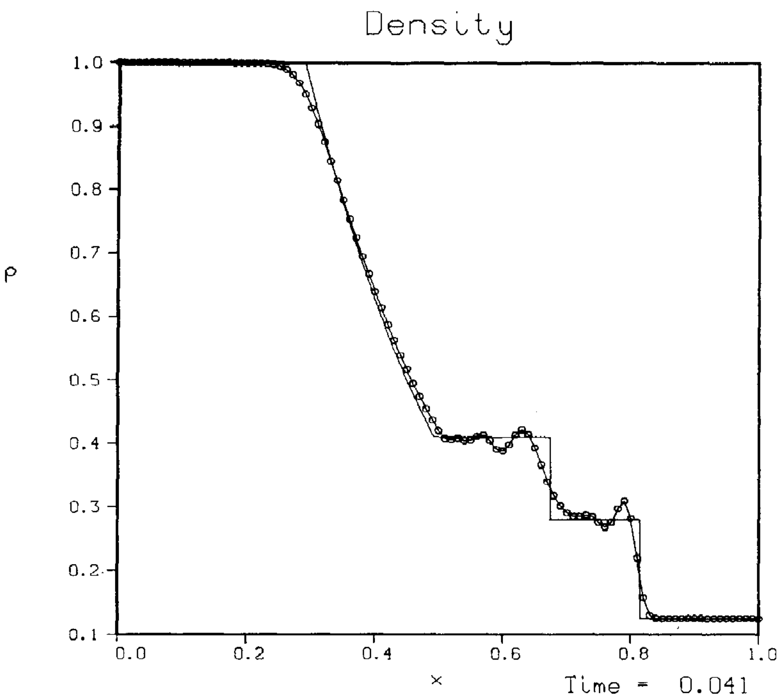


FIG. 1. Shock tube.

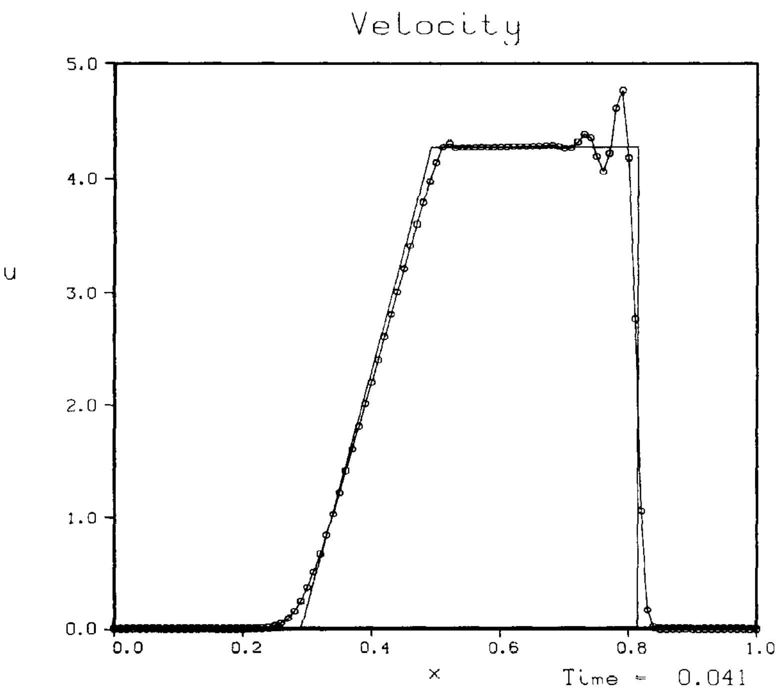


FIG. 2. Shock tube.

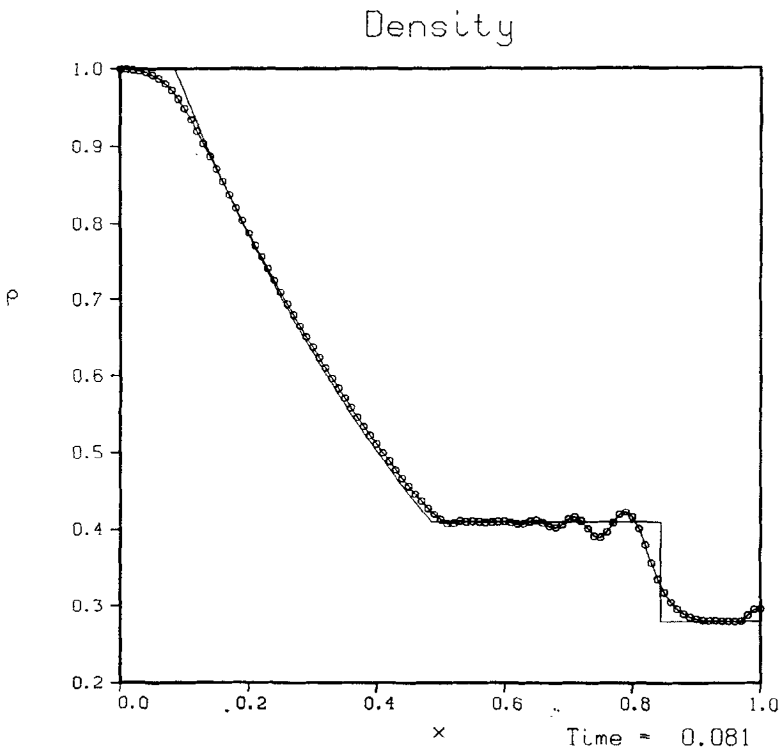


FIG. 3. Shock tube.

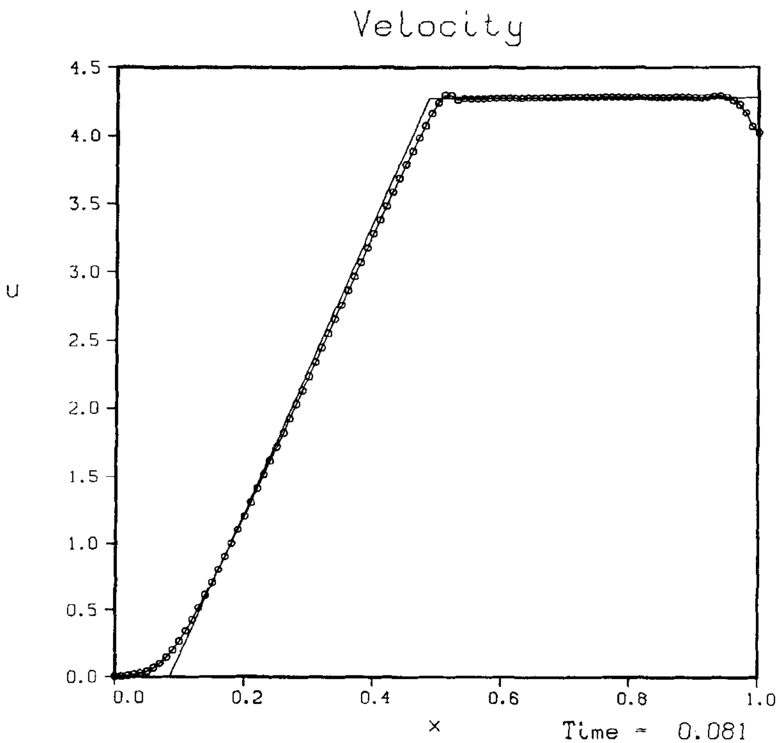


FIG. 4. Shock tube.

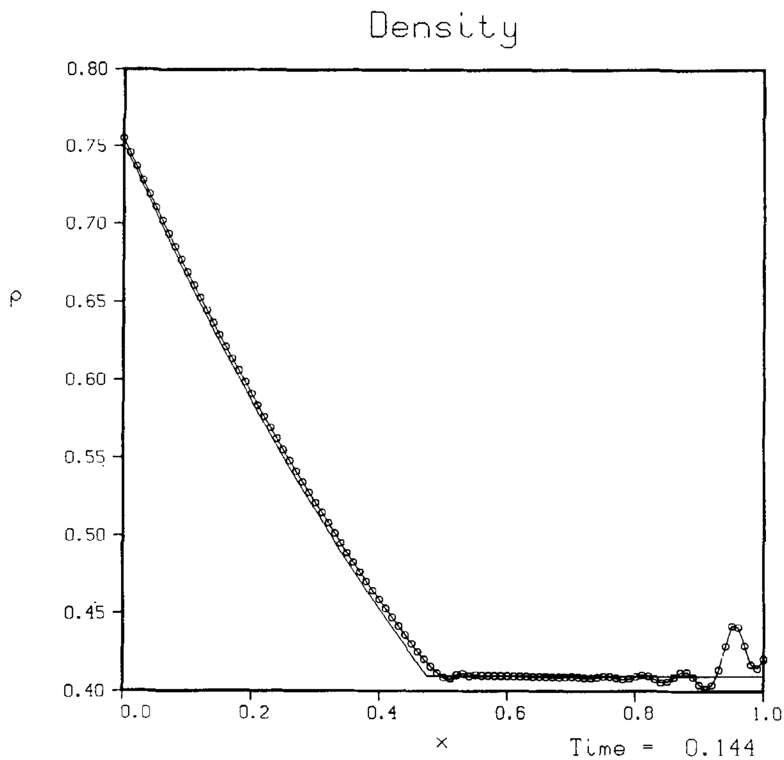


FIG. 5. Shock tube.

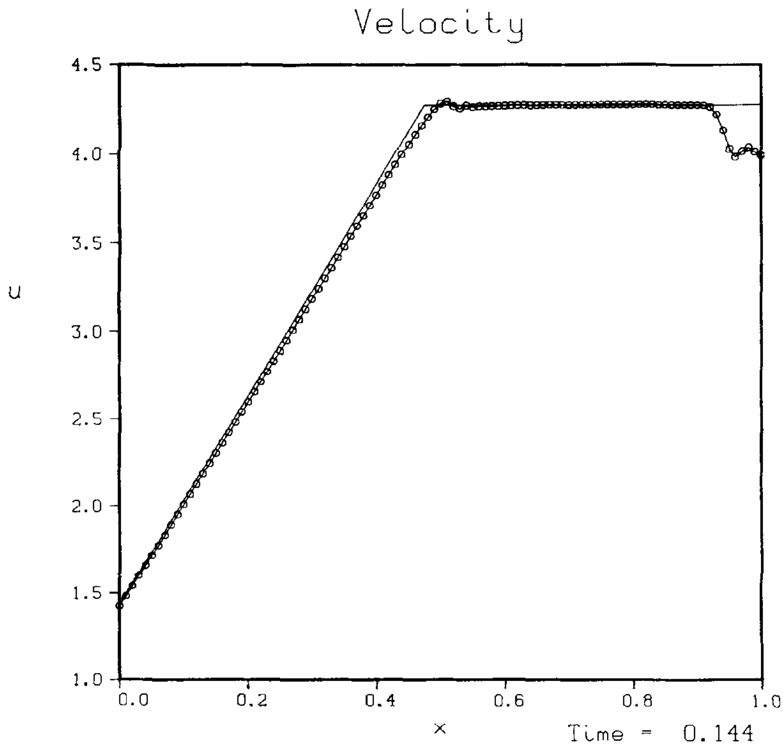


FIG. 6. Shock tube.

duced a weak reflected shock. In Figs. 5 and 6 the contact discontinuity has passed out of the interval and so has part of the rarefaction wave. The reflected shock is now developed. (The apparent increase in the strength of the reflected shck is merely a manifestation of the different scales on the vertical axes.)

Figures 7–12 show the solution of a Riemann problem having two shocks and a contact discontinuity. Initially, the pressure on the right is twice the pressure on the left, so that the shock moving to the right is weaker. In Figs. 9 and 10 the reflected shock on the right is too weak to be visible. It can be seen, though, in Fig. 12 because the scale is magnified. In fact, Fig. 12 shows the two reflected shocks just before they collide. The main feature in Fig. 11 is the contact discontinuity from the original Riemann problem, but we also see a reflected shock and contact discontinuity coming in from the left.

The computations were done on a CDC 7600 computer, and an artificial viscosity of Lapidus type [7] was used.

Finally, we give the strengths of the reflected shocks, measured in terms of the shock Mach number

$$M_s = (\text{shock speed} - u)/(\text{sound speed ahead of shock}).$$

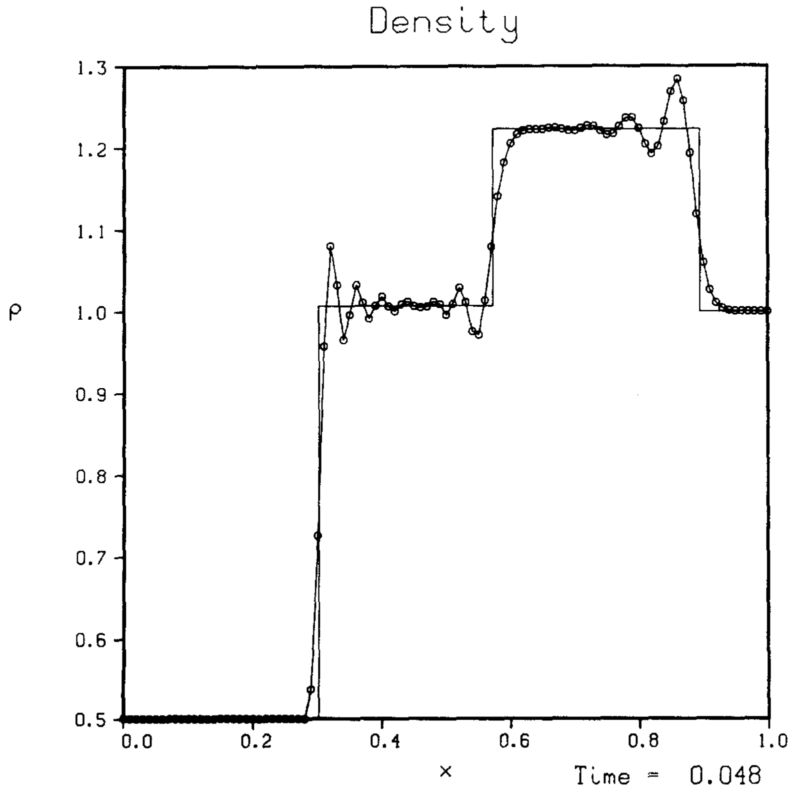


FIG. 7. Two shocks.

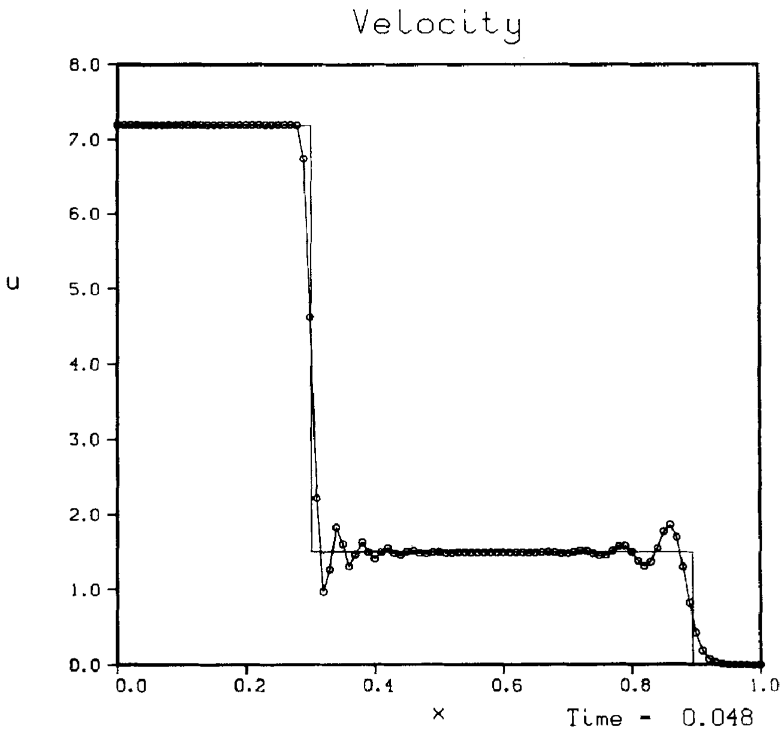


FIG. 8. Two shocks.

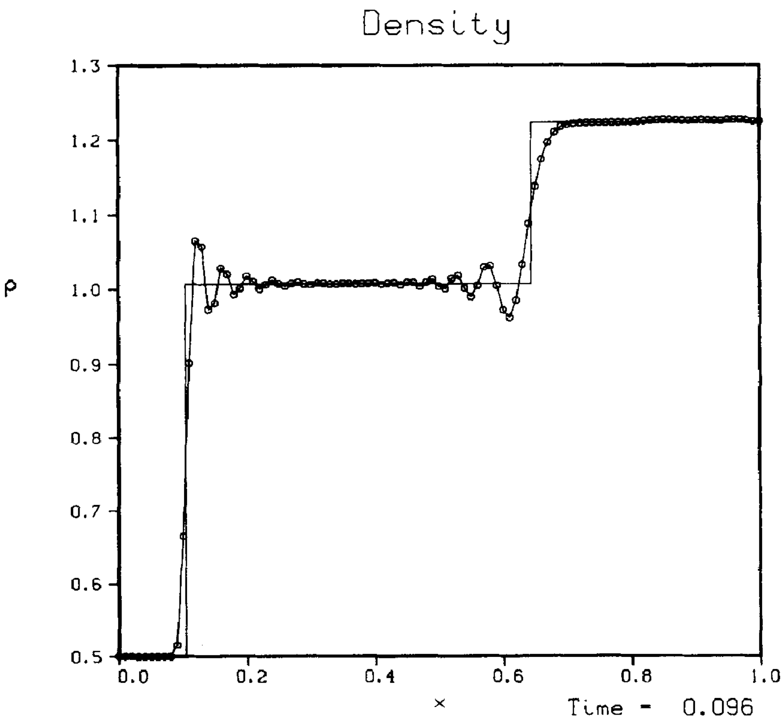


FIG. 9. Two shocks.

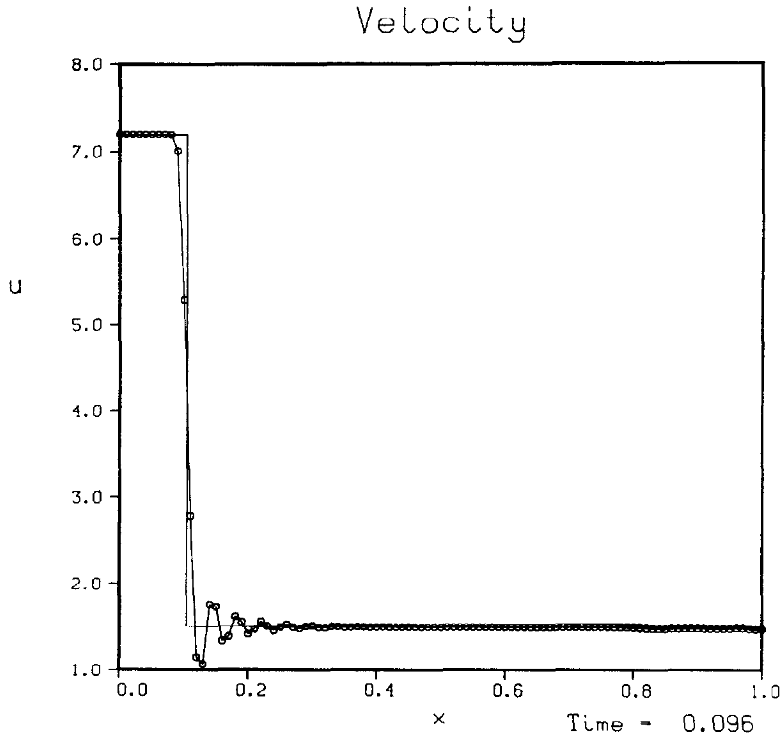


FIG. 10. Two shocks.

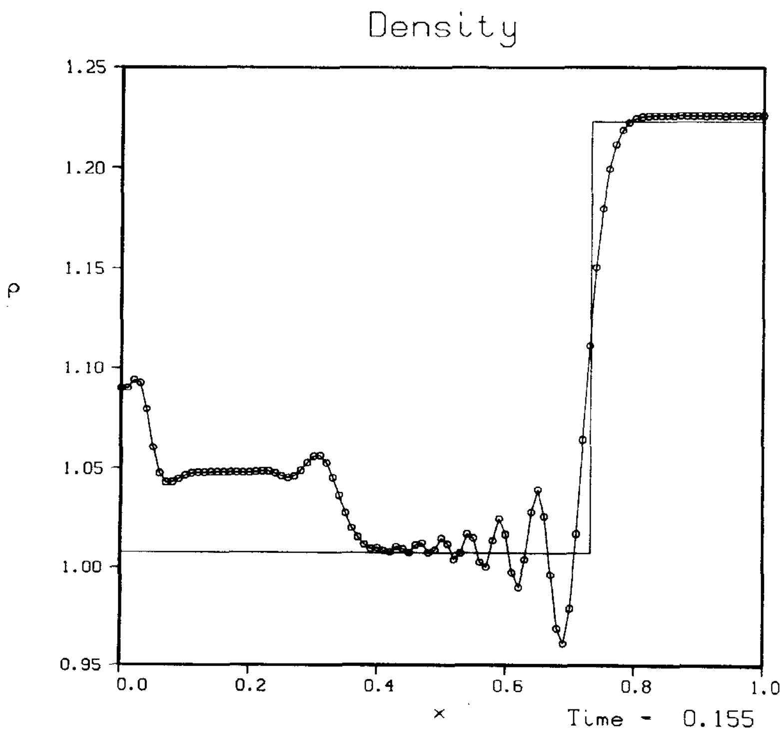


FIG. 11. Two shocks.

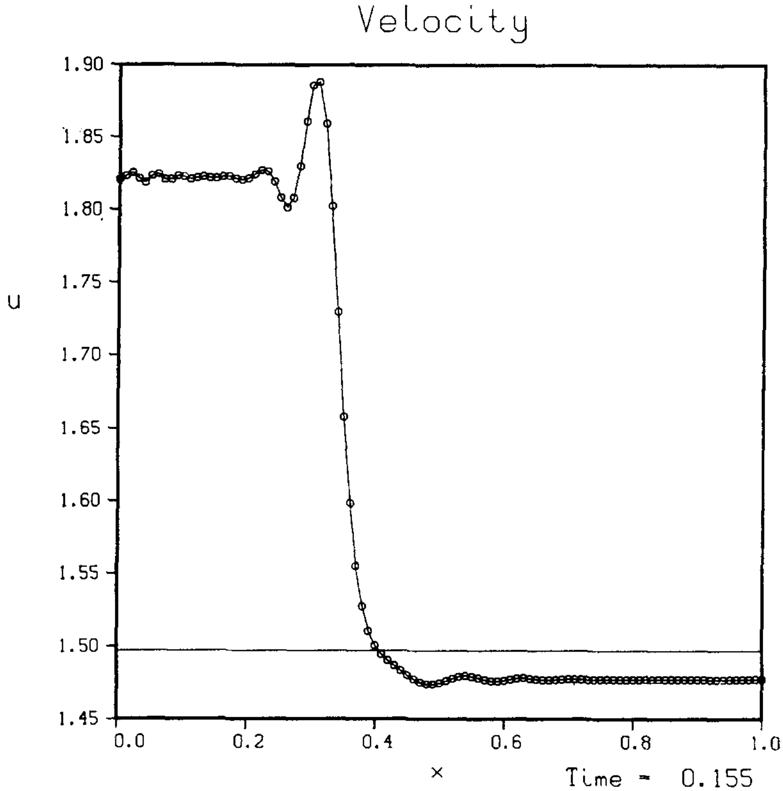


FIG. 12. Two shocks.

For Figs. 1–6 the original shock has $M_s = 1.676$ and the reflected shock has $M_s = 1.038$. For Figs. 7–12 the stronger shock to the left has $M_s = 1.554$ and its reflection has $M_s = 1.023$, while the weaker shock to the right has $M_s = 1.127$ with reflection $M_s = 1.0014$.

## APPENDIX: INTERACTION OF SIMPLE WAVES

Consider the equations of gasdynamics from Section 3. Suppose we have a simple wave associated with $\lambda_1$ overtaking a simple wave associated with $\lambda_2$, that is, a sound wave overtaking a contact discontinuity. If the waves pass through each other, we have the situation shown in Fig. 13. If there is an echo, we have Fig. 14. We shall show that Fig. 14 is the correct one. It is also what is observed in experiments [3, p. 180].

Before the waves meet there are three constant states, state I ahead of the waves, state II between them, and state III behind them. From the discussion in Section 2 we see that states I and II lie on a curve of the family $\Gamma_2$ in state space, and states II and III lie on a curve of the family $\Gamma_1$. Figure 13 would be correct if and only if there

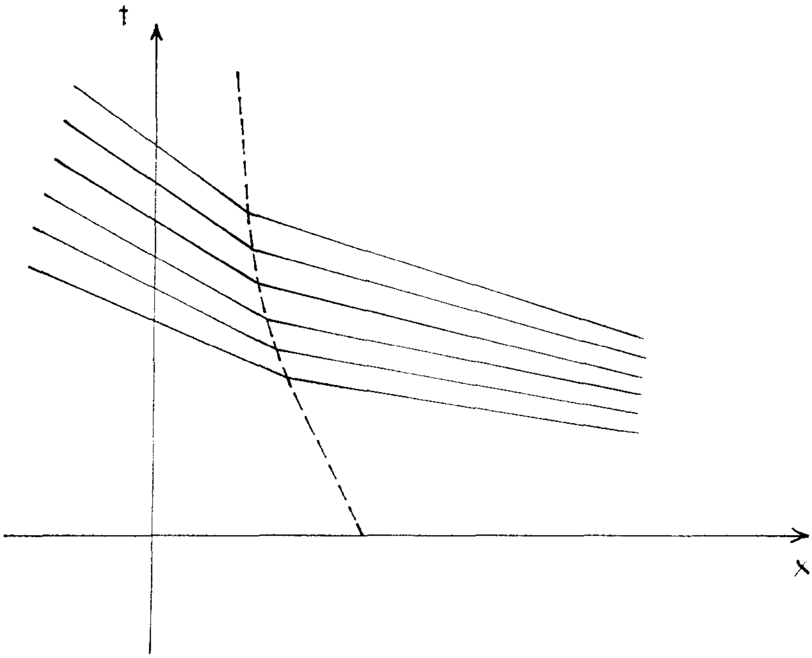


FIG. 13. Incorrect interaction.

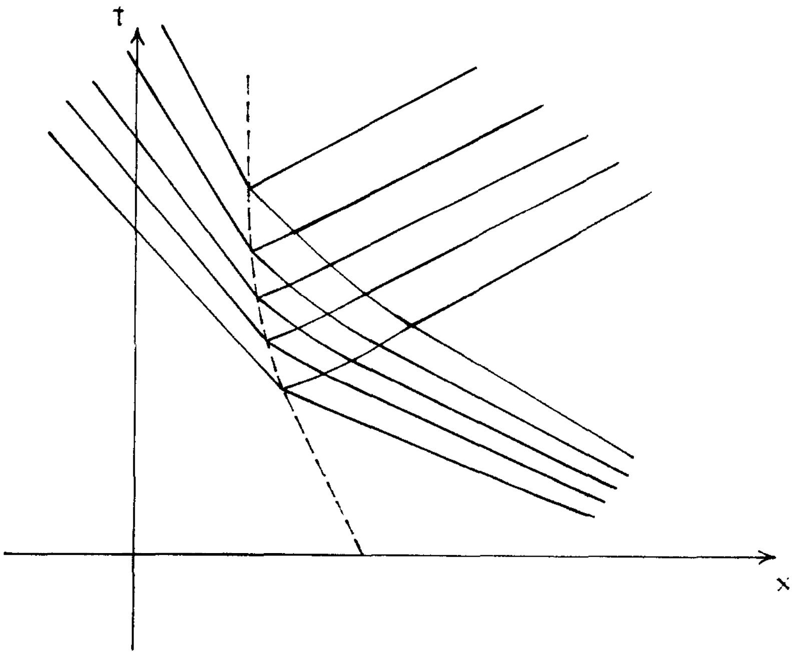


FIG. 14. Correct interaction.

were a state IV connected to state I by a curve of the family $\Gamma_1$ and connected to state III by a curve of the family $\Gamma_2$. In the language of differential geometry we are asserting that the curves $\Gamma_1$ and $\Gamma_2$ do not generate an integrable manifold. In order to decide the question, we look at the 1-form $\omega_3$ associated with the eigenvector $l_3$ of (3.5),

$$\omega_3 = a\,dp + \rho\,du + \rho a/(c_v\gamma)\,dS.$$

(We use $l_3$ because it is orthogonal to $\Gamma_1$ and $\Gamma_2$.) The theorem of Frobenius [8, p. 117] says that $\Gamma_1$ and $\Gamma_2$ generate an integrable manifold if and only if there exists a 1-form $\alpha$ such that

$$d\omega_3 = \omega_3 \wedge \alpha.$$

We now do a routine calculation to show that no such $\alpha$ exists. It follows from (3.3), (3.1), and the rules for differentiating forms [8, pp. 86–89] that

$$\begin{aligned} d\omega_3 &= (a_\rho\,d\rho + a_S\,dS) \wedge dp + dp \wedge du \\ &\quad + (c_v\gamma)^{-1}(a\,d\rho + \rho a_\rho\,d\rho + \rho a_S\,dS) \wedge dS, \\ d\omega_3 &= a/(2\gamma c_v)\,dp \wedge dS + dp \wedge du. \end{aligned}$$

If we take $\alpha$ to be

$$\alpha = \alpha_1\,dp + \alpha_2\,du + \alpha_3\,dS,$$

then we have

$$\begin{aligned} \omega_3 \wedge \alpha = {} & (a\alpha_2 - \rho\alpha_1)\,dp \wedge du \\ & + (a\alpha_3 - \rho a\alpha_1/(c_v\gamma))\,dp \wedge dS + (\rho\alpha_3 - \rho a\alpha_2/(c_v\gamma))\,du \wedge dS. \end{aligned}$$

Hence, $d\omega_3 = \omega_3 \wedge \alpha$ if and only if

$$\begin{aligned} -\rho\alpha_1 + a\alpha_2 &= 1, \\ -\rho a\alpha_1/(c_v\gamma) + a\alpha_3 &= a/(2c_v\gamma), \\ -\rho a\alpha_2/(c_v\gamma) + \rho\alpha_3 &= 0. \end{aligned}$$

A direct Gaussian elimination shows that these equations are inconsistent. Hence, no $\alpha$ exists such that $d\omega_3 = \omega_3 \wedge \alpha$, and no state IV exists to make Fig. 13 correct.

The correct interaction is shown in Fig. 14. Note that the simple waves corresponding to $\lambda_1$ and $\lambda_3$ pass through each other without creating any contact discontinuity. This may be seen either from the fact that the entropy does not change [3, p. 180] or from the fact that by (3.4) we have

$$\omega_2 = dS,$$

so that

$$d\omega_2 = \omega_2 \wedge 0.$$

We conclude by stating the general criterion for determining whether an interaction between simple waves for (1.1) corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ produces waves associated with other eigenvalues. This is just Frobenius' theorem. We first assign to each left eigenvector

$$l_j = (l_{j1}, l_{j2}, \ldots, l_{jn}) \qquad (j > m)$$

a 1-form

$$\omega_j = \sum_k l_{jk}(u)\, du_k .$$

We then compute $d\omega_j$ using the rules

$$d\omega_j = \sum_k \sum_p \left(\frac{\partial l_{jk}}{\partial u_p}\right) du_p \wedge du_k ,$$

$$du_p \wedge du_k = -du_k \wedge du_p ,$$

$$du_k \wedge du_k = 0.$$

Finally, we check by Gaussian elimination whether there exist 1-forms

$$\alpha_{jq} = \sum_k f_{jqk}\, du_k \qquad (m < q \leqslant n)$$

such that

$$d\omega_j = \sum_{q>m} \omega_q \wedge a_{jq} \qquad (m < j \leqslant n).$$

If such $\alpha_{jq}$ exist, no new waves of speeds $\lambda_{m+1}, \ldots, \lambda_n$ are produced. Otherwise, we do get such waves.

## Acknowledgment

This work was performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore Laboratory under Contract W-7405-ENG-48.

## References

1. Fritz John, *Comm. Pure Appl. Math.* **27** (1974), 377.
2. P. D. Lax, *Comm. Pure Appl. Math.* **10** (1957), 227.
3. R. Courant and K. O. Friedrichs, "Supersonic Flow and Shock Waves," Interscience, New York, 1948.
4. Björn Engquist and Andrew Majda, *Math. Comp.* **31** (1977), 629.
5. J. Jeffrey and T. Taniuti, "Non-Linear Wave Propagation," Academic Press, New York, 1964.
6. Robert D. Richtmyer and K. W. Morton, "Difference Methods for Initial-Value Problems," 2nd ed., Interscience, New York, 1967.
7. A. Lapidus, *J. Computational Phys.* **2** (1967), 154.
8. R. Narasimhan, "Analysis on Real and Complex Manifolds," American Elsevier, New York, 1973.